Exam Symmetry in Physics

Date	April 7, 2022
Time	8:30 - 10:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- The weights of the subquestions (a, b, c) of the three exercises are given below
- Illegible answers will not be graded
- Good luck!

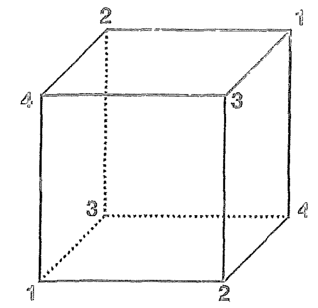
Weighting

1a)	12	2a)	10	3a)	8
1b)	12	2b)	12	3b)	10
1c)	10	2c)	8	3c)	8

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Exercise 1

Consider the cube:



(a) Identify all rotations that leave the cube invariant and show that together they form the group S_4 by looking at the action on the 4 diagonals.

(b) Divide the elements of S_4 into conjugacy classes (using either geometric arguments or the disjoint cycle structure) and determine the dimensions of all the irreps of S_4. The character table does not need to be constructed.

(c) Construct the characters of D^V for the rotational symmetry of the cube and show by explicit calculation that 1) D^V is an irrep of S_4 and that 2) a molecule with the rotational symmetries of the cube cannot have a permanent electric dipole moment.

Exercise 2

Consider the symmetric group S_3 consisting of the permutations of three objects and view the three basis vectors of R^3 as the three objects that are permuted. This leads to the following three-dimensional ($d = 3$) rep D^L of S_3:

$$D^L(c) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D^L(b) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the rep is only specified for the two generators c and b of S_3 which in cycle notation are given by $c = (123)$ and $b = (12)$.

(a) Show that D^L indeed forms a representation of S_3 (for example by using the presentation of the group).

(b) Show that $d^2 > [g]$ implies that D^L is not an irrep of S_3 and decompose D^L into irreps of S_3 using the character table of S_3.

(c) Show that $D^L(g) \in O(3)$ and that D^L is equivalent to the vector rep D^V of the subgroup $C_{3v} \cong S_3$ of $O(3)$, rather than of the subgroup $D_3 \cong S_3$ of $SO(3)$.

Exercise 3

Consider the group $O(2)$ of real orthogonal 2×2 matrices.

(a) Write down explicitly all elements of $O(2)$ in its defining representation.

(b) Consider two vectors in R^2: $\vec{v} = (v_x, v_y)$ and $\vec{w} = (w_x, w_y)$. Demonstrate that the quantity $v_x w_y - v_y w_x$ behaves like a pseudoscalar in R^2.

(c) Derive or deduce the form of the invariant rank-2 tensors T_{ij} that are allowed for an $O(2)$ invariant system (you may use Schur's lemma and assume that the defining representation is an irrep).